

Ivan Treschev

**Discrete Math.
Practice**

*for students
of technical
specialties*

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Discrete Math. Practice. For
students of technical specialties

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Аннотация

The manual is intended for students of the specialty information security of automated systems when studying the practice course “Discrete mathematics”, and can also be used by students of other specialties studying similar courses. The author would like to express gratitude to his supervisor A. Khusainov and Mikhailova N.N.

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Discrete Math. Practice

For students

of technical specialties

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Introduction

This paper presents some of the tasks that can be used in preparation for the Discrete Mathematics course.

The author will be grateful for indicating shortcomings, typos, etc.

1 Examples of completing tasks

Task Problem 1: let A and B be finite sets, $|A| = m$, $|B| = n$. How many binary relations exist between A and B ?

Solution: the Cartesian product will contain a total of $(n \cdot m)$ pairs, therefore, a total of $2^{m \cdot n}$ subsets. Each subset is a binary relation on the set of Cartesian products, which means that there are $2^{m \cdot n}$ relations

Problem No. 2: prove that the comparability relation modulo a positive integer n on the set Z : $x = y \pmod{n}$ is an equivalence relation.

Proof: non the definition of x is comparable with y modulo n if and only if $x - y$ is divisible by n without remainder. It is an equivalence relation, since it holds:

1) Reflexivity – since $x - x = 0$ is divided by n , then $x \equiv x \pmod{n}$.

2) Symmetry – if $x \equiv y \pmod{n}$, then $x - y$ is divided by n , then $y - x$ is divided by n , that is, $y \equiv x \pmod{n}$.

3) Transitivity – if $x - y$ is divisible by n , then $x - y = t_1 n$ (t_1 is an integer), and if $y - z$ is divisible by n , then $y - z = t_2 n$.

Adding these equalities, we have: $x - y + y - z = t_1 n + t_2 n$, whence $x - z = (t_1 + t_2) n$, that is, $x - z$ is divided by n . Thus, from $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n}$ it follows that $x \equiv z \pmod{n}$.

Problem No. 3: a partition of a set X is a collection of pairwise

disjoint subsets of X such that each element of the set X belongs to one and only one of these subsets.

Proof: such a partition, by definition, defines the relation of belonging to a subset of the partition. Therefore, it remains to prove that this relation is equivalence.

1) reflexivity: if for any element x from X xRx performed.

2) Symmetry: Let x, y belong to the same subset, whereas if for all x, y from X in xRy yRx follows.

3) Transitivity: If for any x, y, z from X from xRy , and yRz follows xRz .

A partition of a set X is a collection of pairwise disjoint subsets of X such that each element of the set X belongs to one and only one of these subsets.

Examples.

– $X = \{1, 2, 3, 4, 5\}$. Partition: $\{\{1, 2\}, \{3, 5\}, \{4\}\}$.

– The partition of many students of the institute can be a set of groups.

In other words, a partition of a set X is a family: $X = \cup_{i \in I} X_i \forall i, j$
 $X_i \cap X_j = \emptyset, X_i \subseteq X$ is an element of the partition.

The set of equivalence classes of elements of any set of X by the equivalence relation R sets the quotient of the set X with respect R denoted and X/R . For example, a plurality of student groups of a given university is a factor in a plurality of a plurality of university students in relation to belonging to one group.

Examples.

– $(x, \leq) x \leq y (x, y) \in \leq$ – are comparable;

– $(x, y) \notin \leq$ – are not comparable.

If $(x, \leq), \forall x \in X \exists a \in X | a \leq x$, then a is the smallest element.

If $\exists a \in X | \forall x \in X x \leq a$, then a is the largest element.

Statement. The largest or smallest element is unique.

Evidence. Let a and b be the largest elements in (x, \leq) , then $\forall x \in X a \geq x$, in particular $a \geq b$. Similar to $b \geq a$, therefore $a = b$.

In a similar way, the statement for the smallest elements is proved.

Task number 4: in the cube with side 1, 9 points are selected. Prove that among them there are at least 2 points, the distance between which is $\leq ((\sqrt{3}) \setminus (2))$.

Proof: break a cube into eight identical cubes with side 0.5 – according to the Dirichlet principle, at least one of these cubes will have two points.

The maximum distance between two points in a cube is equal to the size of its diagonal, i.e. $0.5 * \sqrt{3}$.

Problem number 5: in a white sphere, 12% of its area is painted red. To prove that a parallelepiped can be entered into the sphere, in which all vertices are white.

Proof: Draw three mutually perpendicular planes through the center of the sphere and for each point of the sphere we consider its images for symmetries about these planes and for compositions of these symmetries. Each point that does not lie on these planes has exactly 8 images. Therefore, red dots and their images occupy no more than $8.12\% = 96\%$ of the area of the

sphere. Therefore, there is a point for which all 8 images are white. These 8 points are the vertices of a rectangular box.

Problem number 6: prove that among any 10 integers there are several whose sum is divided by 10.

Solution: denote the numbers a_1, a_2, \dots, a_{10} . Among the numbers $0, a_1, a_1+a_2, \dots, a_1+ \dots +a_{10}$ there are two that have the same residues when divided by 10. Then their difference will be divided by 10.

Problem number 7: prove Bernoulli's inequality: if $x \geq -1$, then for $\forall n \in \mathbb{N}$ the following holds: $(1+x)^n \geq 1+nx$.

Solution: we prove the inequality using the method of mathematical induction:

Base of induction: for $n = 1$ we have: $1+x \geq 1+x$ (the inequality holds).

Induction step: let the inequality be true for $k = n$, we prove the inequality for $k = n + 1$: $(1+x)^{n+1} \geq 1 + (n+1)x$

$(1+x)(1+x)^n \geq 1+nx+x$ (1) We use the property: if $a > b$ and $b > c$, then $a > c$, i.e. replace: $(1+x)^n$ with $(1+nx)$: $(1+x)(1+nx) \geq 1+nx+x$

$$1+nx+x+nx^2 \geq 1+nx+x$$

$$nx^2 \geq 0 \quad (2)$$

Since $x^2 \geq 0$ and $n \in \mathbb{N}$, then inequality (2) holds. Bernoulli's inequality is proven.

Problem number 8: prove: $1+3+5+\dots+(2n-1) = n^2$.

Solution: we supplement the left side with even elements up to $2n$, we get the arithmetic progression:

$$1+2+3+4+5+\dots+2n = ((1+2n) / (2)) * 2n = (1+2n) n = n + 2n^2 \quad (1)$$

On the other sides we have: $1+2+3+4+5+\dots+(2n-1) + 2n = 1+3+5+\dots+(2n-1) + 2(1+2+3+\dots+n)$

In the second bracket we see again arithmetic progression, we get:

$$1+3+5+\dots+(2n-1) + 2(1+2+3+\dots+n) = 1+3+5+\dots+(2n-1) + 2((1+n) \setminus (2)) n = 1+3+5+\dots+(2n-1) + (1+n) n = 1+3+5+\dots+(2n-1) + n + n^2 \quad (2)$$

Using (1) and (2) we express $1+3+5+\dots+(2n-1)$: $1+3+5+\dots+(2n-1) = n + 2n^2 - (n + n^2) = n^2$, as required.

This fact can also be proved using the method of mathematical induction.

Problem number 9: prove that $\forall n \in \mathbb{N}$ the inequality holds: $2^n > n$.

Solution: we prove the inequality using the mathematical induction method: Induction

base: for $n = 1$ we have: $2 > 1$.

Induction step: let the inequality be valid for $k = n$, we prove the inequality for $k = n + 1$: $2^{n+1} > n + 1$

$$2 * 2^n > n + 1 \quad (1) \text{ We}$$

use the property: if $a > b$ and $b \geq c$, then $a > c$, i.e. we replace in (1) 2^n with n , we have: $2n \geq n + 1$

$$n \geq 1 \quad (2)$$

Studying inequality (2), we can conclude that $2^n > n$ only for $n \geq 1$, and since $n \in \mathbb{N}$, then this condition is satisfied, the inequality is proved.

Problem number 10: prove the equality: $1^2+2^2+3^2+\dots+n^2 = ((1) \setminus (6)) n (n+1) (2n+1)$.

Solution: we prove the inequality using the mathematical induction method: Induction

base: for $n = 1$ we have: $1 = ((1) \setminus (6)) * 2 * 3 = 1$

Induction step: let the inequality be valid for $k = n$, we prove the inequality for $k = n + 1$: $1^2+2^2+3^2+\dots+n^2+ (n+1)^2 = ((1) \setminus (6)) (n+1) (n+1+1) (2(n+1)+1)$

$$1^2+2^2+3^2+\dots+n^2+ (n+1)^2 = ((1) \setminus (6)) (n+1) (n+2) (2n+3) \quad (1)$$

Replace in (1) $\{1^2+2^2+3^2+\dots+n^2\}$ by $\{((1) \setminus (6)) n (n+1) (2n+1)\}$, we get: $((1) \setminus (6)) n (n+1) (2n+1) + (n+1)^2 = ((1) \setminus (6)) (n+1) (n+2) (2n+3)$

$$n (2n+1) + 6 (n+1) = (n+2) (2n+3)$$

$$2n^2+n+6n+6 = 2n^2+3n+4n+6$$

$$2n^2+7n+6 = 2n^2+7n+6$$

Equality is proved.

Problem number 11: there is a chessboard, a horse is located in the lower left corner. Cut the bottom right corner from the board. Question: is it possible to get around such a chessboard with a knight?

Solution: it is easy to notice that, making a move with the knight, we alternate the color of the cell on which it is located, i.e. if we started with a black cell our movement, then we must finish on white. It is also clear that the left and right lower cells are not the same color. So at the last step we have 2 cells left (a horse is standing on one of them), which we have not yet walked around, and they are of the same color.

Therefore, it is impossible to get around such a chessboard with a horse.

Problem number 12: to prove the inequality: $((1) \setminus (\sqrt{1})) + ((1) \setminus (\sqrt{2})) + ((1) \setminus (\sqrt{3})) + ((1) \setminus (\sqrt{4})) + \dots + ((1) \setminus (\sqrt{n})) + \geq \sqrt{n}$ for $n \geq 0$.

Solution: we prove the inequality using the method of mathematical induction:

Base of induction: for $n = 1$ we have: $1 = 1$.

Induction step: suppose that the inequality is true for $k = n$, we prove the inequality for $k = n + 1$:

$$((1) \setminus (\sqrt{1})) + ((1) \setminus (\sqrt{2})) + ((1) \setminus (\sqrt{3})) + ((1) \setminus (\sqrt{4})) + \dots + ((1) \setminus (\sqrt{n})) + ((1) \setminus (\sqrt{n+1})) \geq \sqrt{n+1} \quad (1) \text{ We}$$

use the property: if $a > b$ and $b \geq c$, then $a > c$ i.e. replace in (1) $((1) \setminus (\sqrt{1})) + ((1) \setminus (\sqrt{2})) + ((1) \setminus (\sqrt{3})) + ((1) \setminus (\sqrt{4})) + \dots + ((1) \setminus (\sqrt{n}))$ by \sqrt{n} , we have: $\sqrt{n} + ((1) \setminus (\sqrt{n+1})) \geq \sqrt{n+1}$

$$\sqrt{n} \geq \sqrt{n+1} - ((1) \setminus (\sqrt{n+1}))$$

$$\sqrt{n} \geq ((n+1) - 1) \setminus (\sqrt{n+1})$$

$$n \geq ((n^2) \setminus (n+1))$$

$$n^2 + n \geq n^2$$

$$n \geq 0 \quad (2)$$

Studying inequality (2), we can conclude that $((1) \setminus (\sqrt{1})) + ((1) \setminus (\sqrt{2})) + ((1) \setminus (\sqrt{3})) + ((1) \setminus (\sqrt{4})) + \dots + ((1) \setminus (\sqrt{n})) + \geq \sqrt{n}$, only for $n \geq 0$, which coincides with the initial conditions, the inequality is proved.

Task number 13: share the booty n robbers. Each of them has their own opinion on the value of a particular share of production, and each of them wants to get no less than $(1/n)$ the share of production (from their point of view). How to divide prey between robbers?

Solution: for two robbers, the task is not difficult to solve – one divides the production into two equal shares in his opinion, and the other selects the largest share from them. We will solve the problem by induction on the number of robbers, i.e. Suppose k robbers already have a way to split prey harmlessly. We will divide prey between $(k + 1)$ robbers. We divide all the prey between k robbers and then let each of them divide his share into $(k + 1)$ equal parts in his opinion. Now let the last robber select one of these parts from each of the k robbers. The last robber took (in his opinion) no less than $[1 / (k + 1)]$ the share of each of the k robbers, i.e. In total, he received at least $[1 / (k + 1)]$ from all production. Each of the first k robbers also received at least $(1 / (k)) * (k / (k + 1)) = (1 / (k + 1))$ from all production.

Problem number 14: prove that an equilateral triangle cannot be covered by two smaller equilateral triangles.

Proof: Each of the smaller triangles cannot cover more than one vertex of a large triangle. According to the Dirichlet principle, there are more cells (in this case, 3 vertices) than rabbits (2 vertices). Therefore, an equilateral triangle cannot be covered by two smaller equilateral triangles.

Problem number 15: prove that $(1+x)^n \geq 1+nx$.

Proof: $(1+x)^{n+1} \geq 1+(n+1)x$

$$(1+x)^n (1+x) \geq (1+nx) + x$$

$$(1+x) (1+x)^n \geq (1+x) (1+nx) + x$$

$$(1+x) (1+nx) = 1+nx+x+nx^2$$

$$1+nx+x \leq 1+nx+x+nx^2$$

That is: $(1+x) (1+x)^n \geq (1+x) (1+nx) \geq (1+nx) + x$

Problem 16 if $(x + ((1) \setminus (x))) -$ an integer, whether integer $x^n + ((1) \setminus (x)) n$?

Proof: we limit the answer by induction. For $n = 0$: $1 + 1 = 2$.
 If $n = -m$, where m is the positive integer: $x^n + ((1) \setminus (x^n)) = x^{-m} + ((1) \setminus (x^{-m})) = x^m + ((1) \setminus (x^m))$

Let $x^k + ((1) \setminus (x^k))$ are integers ($k=0, \dots, k$). Let us prove that $x^{k+1} + ((1) \setminus (x^{k+1}))$ is also an integer: $[(x^k + ((1) \setminus (x^k))) (x + ((1) \setminus (x)))] = x^{k+1} + ((1) \setminus (x^{k+1})) + x^k + ((1) \setminus (x^k)) = [(x^{k+1} + ((1) \setminus (x^{k+1}))) + (x^k + ((1) \setminus (x^k)))] \Rightarrow x^{k+1} + ((1) \setminus (x^{k+1})) = (x^k + ((1) \setminus (x^k))) (x + ((1) \setminus (x))) - (x^k + ((1) \setminus (x^k)))$

The first term on the right-hand side is the product of integers

by the assumption of induction, therefore, the integer, the second is similar. So, the sum is an integer, therefore, the left side is an integer.

Problem number 17: prove that a number made up of 3^n identical digits is divisible by 3^n .

Proof: Fix one of the numbers – a . Denote the number made up of 3^n identical digits a by A_n . We use induction on n . Obviously, A_1 is divisible by 3. Further, suppose that A_k is divisible by 3^k . Note that $A_{k+1} = A_k * 100 \dots 0100 \dots 01$ (the number of zeros in each ellipse is $3^k - 1$). Since the sum of the digits of the number $100 \dots 0100 \dots 01$ is 3, it is divided by 3, and therefore, A_{k+1} is divided by $3^k * 3 = 3^{k+1}$. By induction, the statement of the problem is proved.

Problem number 18: how many ways can decompose the number 1024 into a product of three natural numbers, each of which is greater than 1.

Solution: This is the number of solutions to the equation $x_1 + x_2 + x_3 = 10$, where $x_i > 0$.

Task number 19: a group of 20 students pass tests in mathematics, physics and computer science. Many students who have passed mathematics and physics coincide with many students who have passed mathematics and computer science, and coincide with many students who have passed physics and computer science. Each student has passed at least one test. Find

the number of students who have passed all tests, if they have passed mathematics 15, physics – 16, computer science – 17.

Solution: let M be the set of students who have passed mathematics; F – physics and I – computer science.

According to the inclusion and exclusion formula:

$$|M \cup I \cup \Phi| = |M| + |I| + |\Phi| - |M \cap I| - |M \cap \Phi| - |I \cap \Phi| + |M \cap I \cap \Phi|$$

$$|M| = 15; |I| = 16; |\Phi| = 17;$$

$|M \cup I \cup \Phi| = 20$ (Each student passed at least one test on the tap)

$$|M \cap I| = |M \cap \Phi| = |I \cap \Phi| \Rightarrow |M \cap I| = |M \cap \Phi| = |I \cap \Phi| = |M \cap I \cap \Phi|;$$

$$20 = 15 + 16 + 17 - 2 |M \cap I \cap \Phi|$$

$$2 |M \cap I \cap \Phi| = 15 + 16 + 17 - 20$$

$$|M \cap I \cap \Phi| = ((28) \setminus (2)) = 14$$

So, 14 students passed all the tests.

Answer: 14 students.

Task number 20: 12 knights are sitting at the round table. From them to choose 5 which would not sit nearby.

Solution: we divide the set of all decisions into two subsets, depending on whether a particular knight is in the select team or not. Then we get: $15 + 21 = 36$.

Answer: 36

Problem number 21: 9 camels go in jumble. How many combinations of camel rearrangements exist, in which no camel follows the one he followed.

Solution: select the forbidden pairs: (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9). To solve, we apply the main theorem of combinatorics.

To do this, define what the object is and what the properties are. By objects we mean various arrangements of camels. In total there will be $N = 9!$. By properties we mean the presence of a certain pair in the permutation. Thus, the number of properties is 8. Then the number of permutations that do not have any of the 8 properties:

$$N(8) = 9! - C_8^1 * 8! + C_8^2 * 7! - C_8^3 * 6! + C_8^4 * 5! - C_8^5 * 4! + C_8^6 * 3! - C_8^7 * 2! + C_8^8 * 1! = 142729$$

Answer: 142729.

Task number 22: preference. Cards are laid out in 3 piles and 2 cards are placed in the draw. Play 32 cards, i.e. each player receives 10 cards. Determine the number of layout options.

Solution: using the formula of permutations with repetitions we get: $((32!) \setminus (10!^3 * 2!))$

Answer: $((32!) \setminus (10!^3 * 2!))$.

Task number 23: from a group of 15 people it is necessary to select a team, which should include at least 5 people. How many choices.

Solution: we calculate the number of unfavorable combinations of choice, i.e., we compose options for brigades of 1, 2, 3, 4 people. Their number is: $C_{15}^1 + C_{15}^2 + C_{15}^3 +$

$$C_{15}^4 = 1940.$$

And the total number of brigades is 2^{15} . The difference gives the number of noble combinations.

$$\text{Answer: } 2^{15} - 1914.$$

Task number 24: three boys picked 10 apples. How many ways are there to split apples between them.

Solution: We write 10 units and 2 zeros that perform the functions of a separator, and then we begin to rearrange them with all possible methods. Each permutation will correspond to some way of dividing 10 apples into 3 heaps. Each section method will correspond to some code containing 10 units and 2 zeros. Therefore, the number of partition methods: $P(10,2) = \frac{(10+2)!}{(2!8!)} = 45$

$$\text{Answer: } 45.$$

Problem number 25: a set of 8 elements is necessary to find the number of its partitions.

Solution: The number of partitions of the set is the Bell number $\Rightarrow B_8 = 4140$.

$$\text{Answer: } 4140.$$

Problem number 26: find the number of decimal non-negative integers, consisting of numbers, arranged in increasing order.

$$\text{Solution: } 1 + C_9^1 + \dots + C_9^9 = 2^{10}.$$

$$\text{Answer: } 2^{10}$$

Problem number 27: how many word decompositions from the letters ABRAKADABRA

Solution: in the word ABRAKADABRA there are 11 letters, of which five are letters A, two letters B and two letters R. So, we get by the formula of permutations with repetitions, we get:
 $P_n = ((11!) \setminus (5! * 2! * 2!)) = 83160$

Answer: 83160.

Problem number 28: find the sum of all divisors of 300.

Solution:
$$1+2+3+4+5+6+10+12+15+20+25+30+50+60+75 +100+150 +$$

Answer: 868. This problem can be solved using the Euler formula.

Problem number 29: How many numbers between 1000 and 10000 consist of:

- odd numbers;
- different numbers.

Solution:

a) Between 1000 and 10000 are four-digit numbers. Because numbers consist only of odd numbers, then the first digit can be selected in 5 ways (digits 1, 3, 5, 7 and 9), the second, third and fourth digit can also be selected in 5 ways (digits 1, 3, 5, 7 and 9). Therefore, there will be a total of such numbers:
 $n=5*5 *5 *5=625$

b) numbers consist of various digits, then the first digit can be selected in 9 ways (digits 1, 2, 3, 4, 5, 6, 7, 8 and 9), the second digit in 9 ways (there can be 0 and any of the previous ones, but not repeating), the third digit in 8 ways and the fourth digit in 7 ways. Therefore, there will be a total of such numbers:

$$n=9*9 *8 *7=4536$$

Answer: a) 625; b) 4536.

The task №30: prove $|\sqrt{|x|}| = \sqrt{|x|}$.

Solution: Let $m = |\sqrt{|x|}|$

Because the rules $\leq |x| = n \Leftrightarrow nx < n1$ should $m \leq \sqrt{|x|} < m+1$

$$m^2 \leq |x| < (m+1)^2$$

From the rules: $x < n \Leftrightarrow |x| < n$

$n \leq x \Leftrightarrow n < |x|$, where n an integer

must be $m^2 \leq x < (m+1)^2$

$$m \leq \sqrt{x} < m+1$$

From (*) $\Rightarrow m = |\sqrt{|x|}|$

Thus $|\sqrt{|x|}| = m = |\sqrt{x}|$.

Problem No. 31: $u_{n+2} = -u_{n+1} + 2u_n$, $u_0 = 0$, $u_1 = 1$

Solution: here $K(x) = 1+x-2x^2$. We calculate: $D(x) = K(x)$

$$u(x) = (1+x-2x^2)(u_0 + u_1x + \dots) = 0 + x + 0 = x$$

We get: $u(x) = ((x) \setminus (1+x-2x^2))$

The next step is the expansion of the denominator $K(x)$ to the product $(a_1 - x)(1 - a_2x)$, then $\#_1$ and $\#_2$ – the roots of the quadratic equation: $1+x-2x^2=0$

and

$$a_1 = 1$$

$$a_2 = ((-1) \setminus (2))$$

arrive to the formula: $u(x) = ((x) \setminus (12+x-x^2)) = ((x) \setminus ((1-x)(1+((x) \setminus (2))))))$

We find the decomposition into a sum of simple fractions by the method of indefinite coefficients: $\frac{1}{(1-x)(1+(x/2))} = \frac{A}{(1-x)} + \frac{B}{1+(x/2)}$

We obtain a system of linear equations: $(1) \quad AB=1B+A=0$

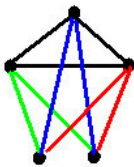
Its solution: $A = \frac{2}{3}, B = \frac{-2}{3}$. Hence: $u(x) = \frac{2}{3} \frac{1}{(1-x)} + \frac{-2}{3} \frac{1}{1+(x/2)} = \frac{2}{3} \sum_{k=0}^{\infty} x^k - \frac{2}{3} \sum_{k=0}^{\infty} (-1)^k x^k$

This leads to the answer: $u_n = \frac{2}{3} - \frac{2}{3} * (-1)^n$

Answer: $u_n = \frac{2}{3} - \frac{2}{3} * (-1)^n$

Task number 32: 15 students shook hands at a meeting of students, three people made 4 handshakes, and others – 3. How many students were there.

Solution: We will consider a case requiring a smaller number of participants. Since the three made 4 handshakes: we consider. That between them they made maximum handshakes – two, and formed a complete 3×3 vertex graph and used only 3 handshakes out of the total:



Each «had» two handshakes. The minimum number of vertices that must be added so that the condition «three made 4 handshakes» is met – these are two. Let's portray them like this:

We get already 9 used handshakes. And the two added vertices have 3 handshakes, which corresponds to the condition. It remains 6. Just the full 4-vertex graph gives us 6 handshakes for each of the 4 students added:

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